On generalized Simes critical constants

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We consider the problem treated by Simes of testing the overall null hypothesis formed by the intersection of a set of elementary null hypotheses based on ordered p-values of the associated test statistics. The Simes test uses critical constants that do not need tabulation. Cai and Sarkar gave a method to compute generalized Simes critical constants which improve upon the power of the Simes test when more than a few hypotheses are false. The Simes constants can be viewed as the first order (requiring solution of a linear equation) and the Cai-Sarkar constants as the second order (requiring solution of a quadratic equation) constants. We extend the method to third order (requiring solution of a cubic equation) constants, and also offer an extension to an arbitrary kth order. We show by simulation that the third order constants are more powerful than the second order constants for testing the overall null hypothesis in most cases. However, there are some drawbacks associated with these higher order constants especially for $k > 3$, which limits their practical usefulness.

Keywords: Multiple hypotheses; Power; Simes test; Type I error.

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1 Introduction

Consider $n \geq 2$ null hypotheses, $H_1, \ldots, H_n$, and denote their associated p-values by $p_1, \ldots, p_n$. Let $p_{(1)} \leq \cdots \leq p_{(n)}$ denote the ordered p-values and $H_{(1)}, \ldots, H_{(n)}$, the corresponding null hypotheses. In this paper we consider the problem of testing the overall null hypothesis $H_0 = \bigcap_{i=1}^{n} H_i$. We assume that the p_i are independent uniform [0, 1] random variables under H_0. The dependence case will be studied in a separate paper.

The Simes (1986) test is based on the identity

$$P\left(\bigcup_{i=1}^{n}\left\{p_{(i)} \leq \frac{i\alpha}{n}\right\}\right) = \alpha, \tag{1}$$

where the probability is computed under H_0 (as are all the type I error probabilities in this paper). Thus it rejects H_0 at level $\alpha \in (0, 1)$ if at least one $p_{(i)} \leq i\alpha/n$ $(1 \leq i \leq n)$. It is more powerful than the Bonferroni test, which rejects H_0 if at least one $p_i \leq \alpha/n$.

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Cai and Sarkar (2008) defined generalized Simes critical constants as any set of c_i $(1 \leq i \leq n)$ that satisfy

$$P\left(\bigcup_{i=1}^{n}\{p_{(i)} \leq c_i\alpha\}\right) = \alpha \tag{2}$$

subject to the monotonicity condition:

$$c_1 \leq \cdots \leq c_n. \tag{3}$$

In this notation, the Simes critical constants are $c_i = i/n$ (note that we use a different notation for critical constants from that used by Cai and Sarkar). The test based on the generalized constants rejects H_0 if

$$p_{(i)} \leq c_i\alpha \quad \text{for at least one } i \quad (1 \leq i \leq n). \tag{4}$$

The monotonicity condition (3) is necessary for this test to be valid as will be seen in the sequel.

In the method given by Cai and Sarkar (2008) to compute these constants, the Simes constants can be viewed as the first order (requiring solution of a linear equation) and the Cai-Sarkar constants as the second order (requiring solution of a quadratic equation) constants. By recursive application of the Cai-Sarkar method we derive third order constants and study their properties in detail. We also present a general result on the kth order constants. Finally, we compare different choices of constants in terms of power via simulation and show that the third order constants improve the power of the test compared to the first and second order constants in a majority of the cases studied.

Bernhard, Klein, and Hommel (2004) have given a nice review of the literature on global and multiple test procedures based on p-values. The following global tests discussed there use special cases of generalized Simes constants. In the case of independent p-values, Bauer (1989) proposed the so-called (n, k, α)-test which uses $c_1 = \cdots = c_{k-1} = 0$ and $c_k = \cdots = c_n = c$ where $c > 0$ is determined from the equation

$$\sum_{i=k}^{n}\binom{n}{i}(c\alpha)^i(1-c\alpha)^{n-i} = \alpha,$$

where k is prespecified. Röhmel and Streitberg (1987) showed that if the p-values are arbitrarily dependent then the α-level is controlled if

$$n\sum_{i=1}^{n}(c_i - c_{i-1})/i \leq 1.$$

The constants that satisfy this condition are (i) Bonferroni: $c_1 = \cdots = c_n = 1/n$, (ii) Rüger (1978): $c_1 = \cdots = c_{k-1} = 0$ and $c_k = \cdots = c_n = k/n$ where k is prespecified and (iii) Hommel (1983): $c_i = i/(n\sum_{j=1}^{n} j^{-1})$.

Generally, a global test does not control the familywise error rate (FWER) if used as a multiple test procedure (MTP). For example, the Simes test does not control the FWER if used to reject any $H_{(i)}$ for which $p_{(i)} \leq i\alpha/n$ $(1 \leq i \leq n)$. An MTP can be derived by constructing a closed procedure (Marcus, Peritz, and Gabriel, 1976) which uses an α-level global test for all intersection hypotheses.

Wei (1996) showed under what conditions this closed procedure has a stepwise shortcut. Toward this end, denote c_i by c_{in} to indicate its dependence on n. Wei (1996) showed that, if the closed procedure uses (4) to test all nonempty subset intersections of H_is of size $m \leq n$ with constants $c_i = c_{im}$, then $c_{im} = c_m$ $(1 \leq i \leq m)$ is a necessary and sufficient condition for the closed procedure to have a step-down shortcut and $c_{im} = c_{i+1,m+1}$ $(1 \leq i \leq m)$ to have a step-up shortcut. The Holm (1979) procedure is the step-down shortcut to a closed procedure that uses the Bonferroni test for all intersection hypotheses. The Hommel (1988) procedure is the closed procedure that uses the Simes test for all intersection hypotheses. But the Simes constants do not satisfy either of Wei's conditions; hence the Hommel procedure does not have a simple stepwise shortcut. Hochberg's (1988) step-up testing procedure can be shown to be based on a conservative choice of the constants, $c_{im} = 1/(m - i + 1)$, which satisfy Wei's condition.

The outline of the paper is as follows. In Section 2 we review the derivation of second order constants. In Section 3 we extend the method to third order constants and study their properties. Section 4 gives a general result about the kth order constants. Section 5 gives tables of the second and third order constants for selected values of c_1 and (c_1, c_2), Section 6 compares the powers of the generalized Simes test for different choices of constants. Conclusions are given in Section 7. Proofs of all the results are given in the Appendix.

2 Second order generalized Simes constants

We assume throughout that the generalized Simes constants satisfy the type I error rate condition (2). Define the probabilities:

$$A_n(i) = \begin{cases} P(p_{(1)} > c_1\alpha, \dots, p_{(n)} > c_n\alpha) & i = 0, \\ P(p_{(i)} \le c_i\alpha, p_{(i+1)} > c_{i+1}\alpha, \dots, p_{(n)} > c_n\alpha) & i = 1, \dots, n-1, \\ P(p_{(n)} \le c_n\alpha) & i = n. \end{cases} \tag{5}$$

Note that $\sum_{i=0}^{n} A_n(i) = 1$ and hence

$$\sum_{i=1}^{n} A_n(i) = 1 - A_n(0) = P\left(\bigcup_{i=1}^{n}\{p_{(i)} \le c_i\alpha\}\right) = \alpha. \tag{6}$$

In the sequel we use a recursion which involves, for fixed n, expressing $A_n(i)$ in terms of $A_{n-1}(i)$, $A_{n-2}(i)$, etc. These lower dimensional probabilities are given by

$$A_{n-m}(i) = \begin{cases} P(p_{(1)} > c_{m+1}\alpha, \dots, p_{(n-m)} > c_n\alpha) & i = 0, \\ P(p_{(i)} \le c_{m+i}\alpha, p_{(i+1)} > c_{m+i+1}\alpha, \dots, p_{(n-m)} \le c_n\alpha) & i = 1, \dots, n-m-1, \\ P(p_{(n-m)} \le c_n\alpha) & i = n-m. \end{cases} \tag{7}$$

Note that when computing $A_{n-m}(i)$ for $n - m < n$, $p_{(1)}$ is compared with $c_{m+1}\alpha$, not with $c_1\alpha$; $p_{(2)}$ is compared with $c_{m+2}\alpha$, not with $c_2\alpha$, etc. The latter would be the case if we change the notation so that the index of c_i is changed from i to $n - i + 1$.

Finner and Roters (1994) showed that under the monotonicity condition (3), the following recurrence relation holds:

$$A_n(i) = \frac{nc_i\alpha}{i} A_{n-1}(i-1) \quad (i = 1, \dots, n). \tag{8}$$

Since this recurrence relation lies at the core of the computation of generalized Simes constants, their validity (in terms of controlling the type I error) requires that the monotonicity condition (3) must be satisfied.

By substituting this recurrence relation in (6) we get

$$\sum_{i=1}^{n} \frac{nc_i\alpha}{i} A_{n-1}(i-1) = \alpha. \tag{9}$$

If we set

$$\frac{nc_i}{i} = \beta_1 \quad (1 \le i \le n), \tag{10}$$

and note that $\sum_{i=1}^{n} A_{n-1}(i-1) = \sum_{i=0}^{n-1} A_{n-1}(i) = 1$ then we obtain from (9) that

$$\alpha\beta_1 \sum_{i=0}^{n-1} A_{n-1}(i) = \alpha\beta_1 = \alpha \implies \beta_1 = 1.$$

Substituting $\beta_1 = 1$ back in (10) yields the Simes constants $c_i = i/n$. Observe that they do not require predetermining any c_is.

Cai and Sarkar (2008) applied the recurrence relation (8) a second time by putting

$$A_{n-1}(i-1) = \frac{n-1}{i-1} c_i \alpha A_{n-2}(i-2) \quad (i = 2, \ldots, n)$$

in (9) to obtain the equation

$$nc_1\alpha + n(n-1)\alpha^2 \sum_{i=2}^{n} \frac{c_i}{i-1}\left(\frac{c_i}{i} - c_1\right) A_{n-2}(i-2) = \alpha. \tag{11}$$

If we set

$$\frac{c_i}{i-1}\left(\frac{c_i}{i} - c_1\right) = \beta_2 \tag{12}$$

and note that $\sum_{i=2}^{n} A_{n-2}(i-2) = 1$, we obtain from (11) that

$$\beta_2 = \frac{1 - nc_1}{n(n-1)\alpha}.$$

Substituting β_2 back in (12) we obtain the quadratic equation:

$$\frac{c_i^2}{i(i-1)} - \frac{c_i c_1}{i-1} - \frac{1 - nc_1}{n(n-1)\alpha} = 0.$$

The roots of this equation depend on α unlike the Simes constants. Furthermore, they depend on c_1, which needs to be specified. Cai and Sarkar (2008) limited the range of c_1 to $0 \le c_1 \le 1/n$ in which case the admissible root is given by

$$c_i = \frac{c_1 i}{2} + \sqrt{\frac{c_1^2 i^2}{4} + \frac{(1 - nc_1)}{\alpha} \frac{i(i-1)}{n(n-1)}}. \tag{13}$$

We can show that the range of c_1 can be extended to $2/[n(1 + \sqrt{1-\alpha})] > 1/n$. However, the second order constants obtained by this extension do not result in any significant power gain. Therefore, we omit the details of this extension. Note that if we put $c_1 = 1/n$ in (13) then we get the Simes constants $c_i = i/n$ and if we put $c_1 = 0$ then we get $c_i = \sqrt{[i(i-1)]/[n(n-1)\alpha]}$.

3 Third order generalized Simes constants

In this section we show how third order constants can be obtained by applying the recurrence relation (8) a second time. Substitute

$$A_{n-2}(i-2) = \frac{n-2}{i-2} c_i \alpha A_{n-3}(i-3) \quad (i = 3, \ldots, n)$$

in (11) to obtain

$$\alpha = nc_1\alpha + n(n-1)c_2\left(\frac{c_2}{2} - c_1\right)\alpha^2 A_{n-2}(0)$$
$$+ n(n-1)(n-2)\alpha^3 \sum_{i=3}^{n} \frac{c_i}{i-2}\left[\frac{c_i^2}{i(i-1)} - \frac{c_i c_1}{i-1}\right] A_{n-3}(i-3). \quad (14)$$

Further substituting

$$A_{n-2}(0) = 1 - \sum_{i=3}^{n} A_{n-2}(i-2)$$
$$= 1 - (n-2)\alpha \sum_{i=3}^{n} \frac{c_i}{i-2} A_{n-3}(i-3)$$

in the second term of (14) and collecting the terms we get

$$\alpha = nc_1\alpha + n(n-1)c_2\left(\frac{c_2}{2} - c_1\right)\alpha^2$$
$$+ n(n-1)(n-2)\alpha^3 \sum_{i=3}^{n} \frac{c_i}{i-2}\left[\frac{c_i^2}{i(i-1)} - \frac{c_i c_1}{i-1} - \frac{c_2^2}{2} + \frac{c_1 c_2}{1}\right] A_{n-3}(i-3).$$

Now set

$$\frac{c_i}{i-2}\left[\frac{c_i^2}{i(i-1)} - \frac{c_i c_1}{i-1} - \frac{c_i c_2^2}{2} + \frac{c_1 c_2}{1}\right] = \beta_3 \quad (15)$$

in the above equation and use the fact that $\sum_{i=3}^{n} A_{n-3}(i-3) = 1$ to obtain

$$\alpha = nc_1\alpha + n(n-1)c_2\left(\frac{c_2}{2} - c_1\right)\alpha^2 + n(n-1)(n-2)\alpha^3\beta_3.$$

Solving this equation we get

$$\beta_3 = \frac{1}{\alpha^2(n-2)}\left[\frac{1}{n(n-1)} - \frac{c_1}{n-1} - c_2\alpha\left(\frac{c_2}{2} - c_1\right)\right].$$

Substituting this value of β_3 back in (15) we obtain the following cubic equation for c_i:

$$f_i(c_i) = c_i^3 + q_i c_i^2 + r_i c_i + s_i = 0, \quad (16)$$

where

$$q_i = -ic_1,$$
$$r_i = -i(i-1)c_2\left(\frac{c_2}{2} - c_1\right), \quad (17)$$
$$s_i = -\frac{i(i-1)(i-2)}{\alpha^2(n-2)}\left[\frac{1}{n(n-1)} - \frac{c_1}{n-1} - c_2\alpha\left(\frac{c_2}{2} - c_1\right)\right].$$

Note that the solutions of the cubic Eq. (16) depend on α as well as (c_1, c_2), which must be specified. The following special cases are worth noting:

(1) If we use the same (c_1, c_2) for the third order constants as for the second order constants with

$$c_2 = c_1 + \sqrt{c_1^2 - \frac{2}{\alpha(n-1)}\left(c_1 - \frac{1}{n}\right)}, \tag{18}$$

from (13), then the third order constants $c_3, \ldots, c_n$ obtained by solving the cubic Eq. (16) are the same as the second order constants.

(2) As a special case of the above, if we put $c_1 = 1/n$ and $c_2 = 2/n$, then the solutions to the cubic Eq. (16) are $c_i = i/n$ $(3 \le i \le n)$, which are the Simes constants. This result extends the corresponding result for second order constants where, if we put $c_1 = 1/n$, then we get the Simes constants.

(3) If we put $c_1 = c_2 = 0$ then the solution is

$$c_i = \sqrt[3]{\frac{i(i-1)(i-2)}{\alpha^2 n(n-1)(n-2)}} \quad (3 \le i \le n). \tag{19}$$

We now state the main theorem about the third order constants.

Theorem 1. *If the following conditions hold:*

(1) $\alpha \le \frac{n}{2(n-1)}$,
(2) $0 \le c_1 \le \frac{1}{n}$,
(3) $2c_1 \le c_2 \le c_1 + \sqrt{c_1^2 + \frac{2}{\alpha(n-1)}\left(\frac{1}{n} - c_1\right)}$,

then the cubic Eq. (16) *has a unique positive root* c_i *and* $c_1, \ldots, c_n$ *satisfy the monotonicity condition* (3). *Furthermore, all* $c_i \le \alpha^{-2/3} < 1/\alpha$ *so that the* p_i*s are compared with* $c_i\alpha < 1$.

Remark 1. The reasons for the three conditions are as follows. First, $\alpha \le n/2(n-1)$ holds because generally $\alpha \le 1/2$. Second, the range of c_1 can be extended to

$$0 \le c_1 \le \frac{1}{\alpha}\sqrt{\frac{1}{n-1}}\left(\sqrt{\frac{1}{n-1}} - \sqrt{\frac{1}{n-1} - \frac{2\alpha}{n}}\right),$$

where the upper limit is $> 1/n$. Similarly, the lower bound on c_2 can be extended to

$$c_2 > \left(1 + \sqrt{\frac{n-3}{3(n-1)}}\right)c_1,$$

which is $<2c_1$; see (A.1) in the proof of Result 1 in the Appendix. However, if we extend these ranges, the cubic Eq. (16) does not have a unique positive root, which poses difficulties in choosing the particular positive root and showing the monotonicity of the chosen set of roots c_i $(3 \le i \le n)$. This point will become clearer in the proof of Theorem 1 given in the Appendix.

Cai and Sarkar (2008) showed that if $\alpha < [(i-1)n]/[i(n-1)]$ then the second order constants c_i are decreasing functions of c_1 for each $i = 2, \ldots, n$. The following theorem gives an extension of this result for the third order constants:

Theorem 2. *If* $\alpha \le [n(n-1)]/[6(n-2)^2]$ *and* $c_1 \in [0, 1/n]$ *is fixed then the* c_i *are decreasing functions of* c_2 *for each* $i = 3, \ldots, n$ *for*

$$c_2 \in \left[2c_1, c_1 + \sqrt{c_1^2 + \frac{2}{\alpha(n-1)}\left(\frac{1}{n} - c_1\right)} \right]. \tag{20}$$

Furthermore, if $c_1, c_2, \ldots, c_n$ *are third order constants and* $c_1', c_2', \ldots, c_n'$ *are second order constants such that* $c_1 = c_1'$ *and* c_2 *is strictly less than the upper limit of the interval* (20) *then* $c_2 < c_2'$ *and* $c_i > c_i'$ *for* $i = 3, \ldots, n$.

Note that the upper bound on α in this theorem equals 1 for $n = 3$ and is a decreasing function of n, equaling 0.3704 for $n = 5$, 0.2344 for $n = 10$ and approaching 1/6 as $n \to \infty$. Thus this theorem holds for all practical values of α and n. As a result of this theorem, given any set of second order constants, we can find third order constants such that they are larger for $i = 3, \ldots, n$, which would make them more powerful in many cases. We will illustrate this numerically in Sections 5 and 6.

4 *k*th order generalized Simes constants

We can apply the successive recursion process employed in the previous section $k - 1$ times to obtain the kth order constants for any $k < n$. They require one to specify the first $k - 1$ critical constants from which the remaining ones can be determined by solving a kth degree polynomial equation. As such, this generalization is not of much practical use but we give it here for theoretical interest.

Theorem 3. *In general, one can determine the constants* $c_1 \le \cdots \le c_n$ *(where* $c_1 \le 1/n$*), which satisfy the type I error requirement* (2) *by specifying* $c_1, \ldots, c_{k-1}$ *subject to certain constraints and then by solving for* c_{k+i} $(0 \le i \le n-k)$ *from the kth degree polynomial equation:*

$$\beta_k(i) + \frac{\alpha - \gamma_k}{\alpha^k} = 0,$$

where $\beta_k(i)$ *and* γ_k *are defined recursively by the following set of equations: Let*

$$\delta(i, n) = \frac{nc_i}{i}, \ \beta_1(i) = \delta(i+1, n) \ (i = 0, \ldots, n-1) \ \text{ and } \ \gamma_1 = 0.$$

Then

$$\beta_{k+1}(i) = [\beta_k(i+1) - \beta_k(0)]\delta(i+1, n-k) \quad (i = 0, \ldots, n-k-1)$$

and

$$\gamma_{k+1} = \gamma_k + \alpha^k \beta_k(0).$$

5 Tables of generalized Simes constants

In this section we give the second and third order generalized Simes constants for $\alpha = 0.05$ and $n = 3, 4, 5$ in Tables 2, 3, and 4, respectively. The Simes constants $c_i = i/n$, which are independent of α, are also included for comparison purposes and are tabulated under Column I. The other six columns list two choices of second order constants (under columns labeled II, III) and four choices of third order constants (under columns labeled IV–VII). These choices are shown as labeled points I–VII in the admissible region of (c_1, c_2) for third order constants shown in Fig. 1. The upper boundary of this region gives the admissible values of (c_1, c_2) for second order constants.

Table 1 Constants (c_1, c_2) for seven choices.

No.	Type	c_1	c_2
I	Simes	$\frac{1}{n}$	$\frac{2}{n}$
II	Second order	0	$\sqrt{\frac{2}{\alpha n(n-1)}}$
III		$\frac{1}{2n}$	$\frac{1}{\sqrt{2n}}\left(\frac{1}{\sqrt{2n}}+\sqrt{\frac{1}{2n}+\frac{2}{\alpha(n-1)}}\right)$
IV	Third order	0	0
V		0	$\sqrt{\frac{1}{2\alpha n(n-1)}}$
VI		$\frac{1}{2n}$	$\frac{1}{n}$
VII		$\frac{1}{3n}$	$\frac{2}{3\sqrt{n}}\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{2\alpha(n-1)}}\right)$

Table 2 Generalized Simes constants $(n = 3, \alpha = 0.05)$.

c_i	Simes	Second order		Third order			
	I	II	III	IV	V	VI	VII
c_1	0.333	0.000	0.167	0.000	0.000	0.167	0.111
c_2	0.667	2.582	2.000	0.000	1.291	0.333	1.083
c_3	1.000	4.472	3.422	7.368	6.943	6.020	6.224

Table 3 Generalized Simes constants $(n = 4, \alpha = 0.05)$.

c_i	Simes	Second order		Third order			
	I	II	III	IV	V	VI	VII
c_1	0.250	0.000	0.125	0.000	0.000	0.125	0.083
c_2	0.500	1.826	1.422	0.000	0.913	0.250	0.775
c_3	0.750	3.162	2.431	4.642	4.415	3.813	3.959
c_4	1.000	4.472	3.422	7.368	6.943	6.020	6.219

Table 4 Generalized Simes constants $(n = 5, \alpha = 0.05)$.

c_i	Simes	Second order		Third order			
	I	II	III	IV	V	VI	VII
c_1	0.200	0.000	0.100	0.000	0.000	0.100	0.067
c_2	0.400	1.414	1.105	0.000	0.707	0.200	0.605
c_3	0.600	2.450	1.889	3.420	3.268	2.818	2.931
c_4	0.800	3.464	2.658	5.429	5.135	4.446	4.601
c_5	1.000	4.472	3.422	7.368	6.943	6.020	6.217

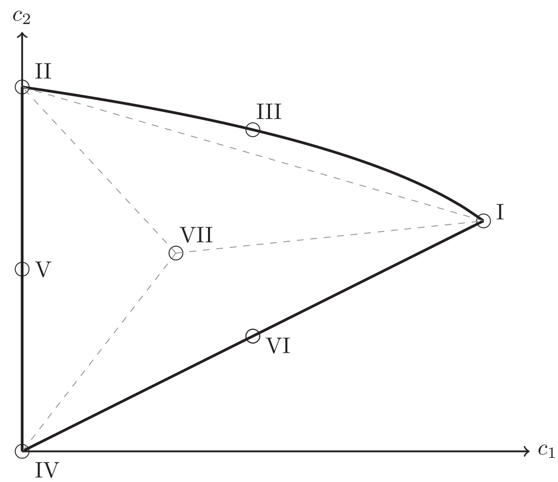

Figure 1 Feasible region of (c_1, c_2).

These choices were determined as follows. First, I, II, and IV are the corner points of the admissible region shown in Fig. 1. Next III, V, and VI are the midpoints of the three boundaries of the region. Finally, VII is the centroid of the triangle formed by the vertices I, II, and IV. The (c_1, c_2)-values for columns I–VII are listed in Table 1.

Note that the upper bound on all the second order constants can be shown to be $\alpha^{-1/2} = 4.472$, which is achieved for c_n under column II when $c_1 = 0$. In Theorem 1, the upper bound on all the third order constants can be shown to be $\alpha^{-2/3} = 7.368$ which is achieved for c_n under column IV when $c_1 = c_2 = 0$.

6 Simulation comparisons

Since all constants satisfy (2), they all control the type I error. So we focus on comparing their powers. Note that here power is simply the probability of rejecting $H_0 = \bigcap_{i=1}^{n} H_i$ when at least one H_i is false. Let m be the number of false null hypotheses. We studied the following configurations: $n = 10$ null hypotheses, $\alpha = 0.05$, and $m = 2(2)10$. For each configuration we made a total of 10^9 simulation runs. In each run we generated $n - m$ values of $N(0, 1)$ and m values of $N(\delta_i, 1)$ random variates where the means δ_i were chosen in two different ways:

1. Constant Means Configuration: $\delta_i = \delta$ $(1 \leq i \leq m)$ where $\delta = 0.5, 1.0, 1.5$.
2. Linear Means Configuration: $\delta_i = i\gamma$ $(1 \leq i \leq m)$ where the slope $\gamma = 2\delta/(m + 1)$ and $\delta = 0.5$, 1.0, 1.5.

The slope for the linear means configuration is chosen so that the average of the δ_is is δ, the same as for the constant means configuration case. Next we transformed the normal variates to p-values and then used these same set of p-values to test H_0 with different choices of constants. The Simes constants (choice I) were used as the basis for comparison. The simulated powers for the constant configuration case are given in Table 5 and those for the linear means configuration case are given in Table 6.

The differences in powers of the six choices, II through VII, of generalized Simes constants with respect to the Simes constants are plotted in Fig. 2 and in Fig. 3 for constant and linear means configurations, respectively, for the remaining six choices of constants as bar charts with the bars labeled as II–VII. The Simes power is noted at the top of each bar chart.

The following conclusions can be drawn from these bar charts.

Table 5 Power comparison (constant means configuration, $n = 10$, $\alpha = 0.05$).

m	δ	Simes	Second order		Third order			
		I	II	III	IV	V	VI	VII
2	0.5	0.077	0.084	0.083	0.086	0.087	0.085	0.087
	1.0	0.151	0.158	0.167	0.151	0.159	0.167	0.170
	1.5	0.300	0.290	0.328	0.241	0.277	0.315	0.321
4	0.5	0.104	0.127	0.121	0.137	0.137	0.129	0.133
	1.0	0.245	0.314	0.303	0.324	0.335	0.322	0.334
	1.5	0.494	0.605	0.599	0.594	0.626	0.618	0.635
6	0.5	0.131	0.176	0.162	0.200	0.199	0.180	0.188
	1.0	0.331	0.476	0.442	0.524	0.529	0.493	0.511
	1.5	0.642	0.818	0.790	0.848	0.858	0.835	0.849
8	0.5	0.157	0.231	0.206	0.272	0.269	0.239	0.249
	1.0	0.410	0.622	0.569	0.701	0.699	0.651	0.670
	1.5	0.751	0.927	0.901	0.957	0.958	0.943	0.950
10	0.5	0.183	0.288	0.252	0.351	0.344	0.302	0.317
	1.0	0.482	0.741	0.679	0.830	0.824	0.778	0.795
	1.5	0.831	0.974	0.958	0.990	0.990	0.984	0.986

m = number of false hypotheses

Table 6 Power comparison (linear means configuration, $n = 10$, $\alpha = 0.05$).

m	δ	Simes	Second order		Third order			
		I	II	III	IV	V	VI	VII
2	0.5	0.080	0.085	0.085	0.087	0.088	0.087	0.088
	1.0	0.171	0.161	0.182	0.149	0.159	0.180	0.181
	1.5	0.358	0.282	0.371	0.227	0.264	0.356	0.354
4	0.5	0.114	0.133	0.130	0.139	0.141	0.136	0.140
	1.0	0.307	0.343	0.359	0.327	0.351	0.365	0.374
	1.5	0.633	0.646	0.702	0.567	0.640	0.694	0.707
6	0.5	0.147	0.189	0.179	0.206	0.208	0.194	0.202
	1.0	0.429	0.535	0.529	0.543	0.566	0.557	0.574
	1.5	0.801	0.873	0.884	0.850	0.886	0.896	0.905
8	0.5	0.180	0.251	0.231	0.285	0.285	0.260	0.271
	1.0	0.534	0.696	0.672	0.731	0.745	0.722	0.739
	1.5	0.896	0.963	0.961	0.964	0.974	0.973	0.977
10	0.5	0.213	0.317	0.286	0.370	0.368	0.331	0.345
	1.0	0.622	0.815	0.782	0.861	0.866	0.841	0.855
	1.5	0.948	0.991	0.989	0.994	0.995	0.994	0.995

m = number of false hypotheses.

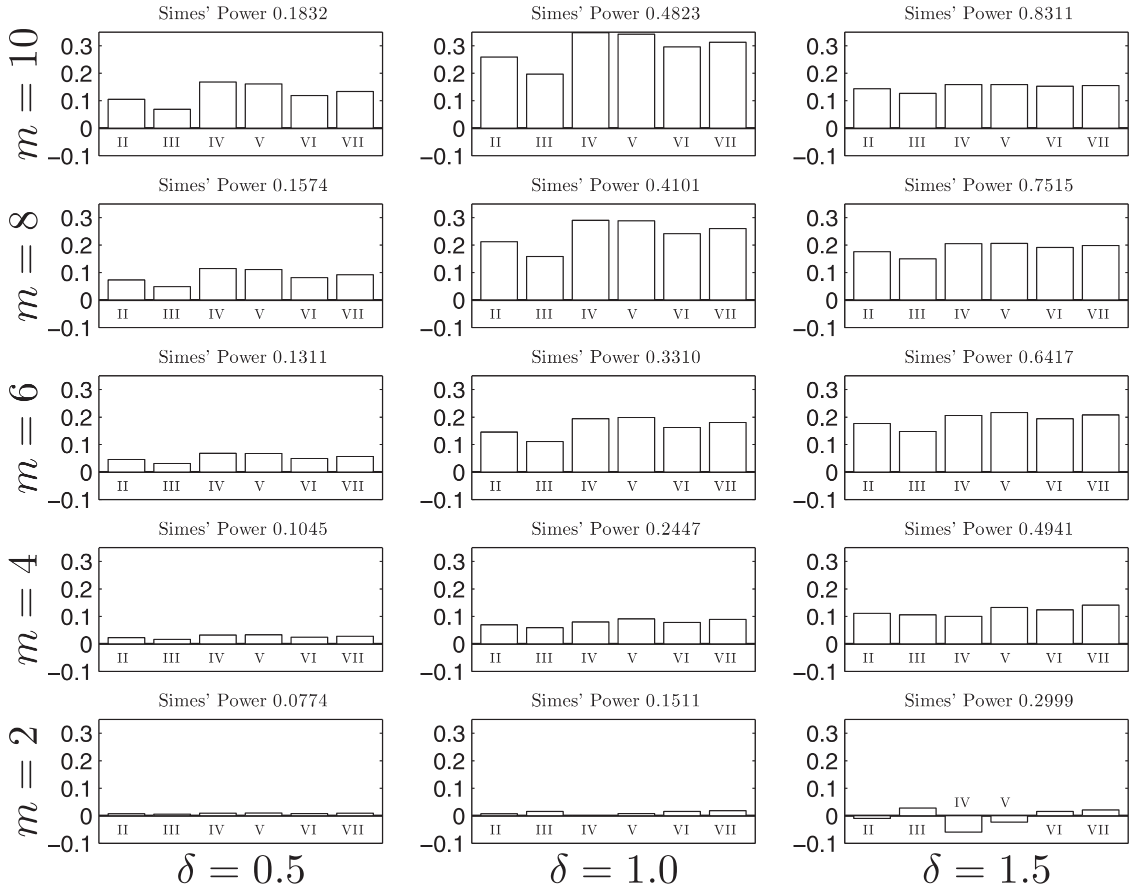

Figure 2 Power gains of second and third order generalized Simes critical constants over the first order Simes critical constants (constant δ_i configuration).

1. The original Simes constants compare favorably in power with higher order constants only when $m = 2$ hypotheses are false. This result agrees with that observed by Cai and Sarkar for second order constants.
2. For each fixed δ, the power gains of both the second order and third order constants increase as the number of false null hypotheses increases.
3. Third order constants generally yield higher powers than second order constants.
4. Maximum power gains by second order and third order constants are attained at $\delta = 1.0$. This is natural since as δ decreases, all powers approach α and as δ increases, all powers approach 1. So the maximum power gains are achieved at a medium value of δ.
5. Generally, choices IV and V have the highest power gains but they are less powerful than the Simes constants when $m = 2$ and $\delta = 1.5$, so they are recommended in other cases. On the other hand, choices VI and VII have uniformly high power gains (although not always the highest) in all cases, and are thus robust to unknown number of false hypotheses, with choice VII beating choice VI in all cases. Thus choice VII, which is approximately the centroid of the admissible region of (c_1, c_2), is recommended.

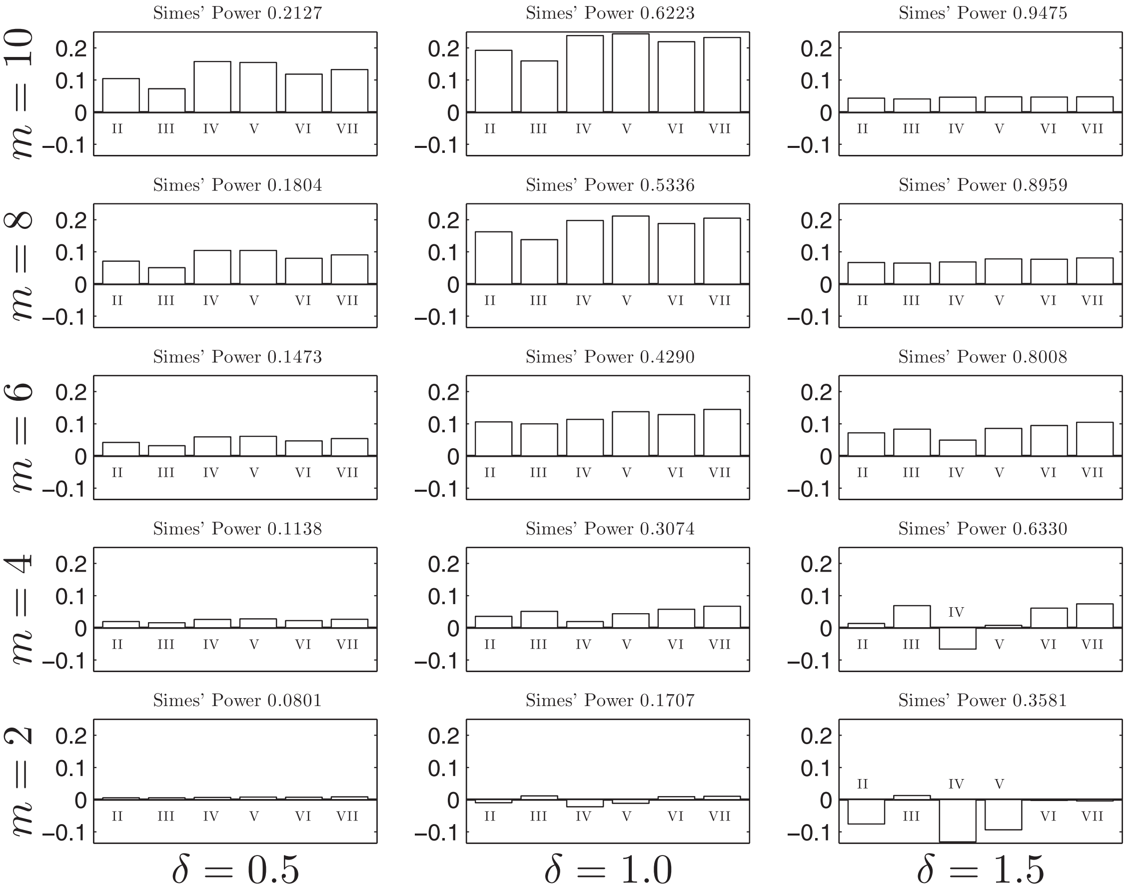

Figure 3 Power gains of second and third order generalized Simes critical constants over the first order Simes critical constants (linear trend δ_i configuration).

7 Concluding remarks

We have shown how higher order generalized Simes constants can be derived and computed. The original Simes constants compare favorably in power with higher order generalizations only when a few hypotheses are false. When more hypotheses are false both the second order and third order constants are significantly more powerful with the third order constants being more so.

Although not reported here due to space constraints, we also made power comparisons between Bauer's (1989) (n, k, α)-test and the generalized Simes test and found that the latter provides a more powerful test. The details are available from the authors.

Associated with these higher powers there are also some drawbacks: First, one needs to specify c_1 for second order constants and (c_1, c_2) for third order constants (more generally, $(c_1, \dots, c_{k-1})$ for the kth order constants for $k < n$). Second, the kth order constants require solving a kth degree polynomial equation and choosing a suitable positive root satisfying the monotonicity condition which is not easy.

All the comparisons in this paper are restricted to the independence case. We have made some preliminary simulation studies under dependence which suggest that the higher order constants control the type I error under negative dependence but not under positive dependence. This is opposite of the behavior of the Simes constants (Samuel-Cahn 1996, Sarkar and Chang 1997, Sarkar 1998). We have developed a method to robustify the second and third order constants so that they approximately control the type I error, while still achieving substantial power gains over the Simes constants. We will report these developments in a separate paper.

Finally, one could obtain MTPs by applying the closure method using these generalized Simes constants. These MTPs will not have a step-up shortcut since the constants do not satisfy Wei's (1996) condition; however they will be more powerful than the Hochberg, Hommel or Rom MTPs.

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Conflict of interest
The authors have declared no conflict of interest.

Appendix

Proof of Theorem 1. The proof is in a number of parts stated as Results. For compactness of notation we will denote the cubic $f_i(c_i)$ defined in (16) by $f(x)$, dropping the subscript i from $f_i(x)$, q_i, r_i, and s_i until needed in the final part of the proof. The proof involves studying the critical points (where the derivative $f'(x)$ of $f(x)$ is zero) and roots of $f(x)$ (where $f(x) = 0$). Basically, we show that under the three conditions stated in the theorem, $f(x)$ has only one positive root and $f(0) \leq 0$. The possible shapes of $f(x)$ are shown in Fig. A1. We want to rule out the cases (a) and (b). □

Result 1. When $c_2 \geq 2c_1$, the cubic $f(x)$ has two real critical points corresponding to the four cubic curves shown in Fig. A1.

Proof of Result 1. The derivative of $f(x)$ is

$$f'(x) = 3x^2 + 2q + r = 3x^2 - 2ic_1x - i(i-1)c_2\left(\frac{c_2}{2} - c_1\right).$$

The discriminant of this quadratic is

$$\begin{aligned}\Delta &= 4i\left[ic_1^2 + 3(i-1)c_2\left(\frac{c_2}{2} - c_1\right)\right] \\ &= 2i\left[3(i-1)(c_2 - c_1)^2 - (i-3)c_1^2\right].\end{aligned}$$

Then $f'(x) = 0$ will have two real roots if and only if

$$\begin{aligned}\Delta > 0 &\iff 3(i-1)(c_2 - c_1)^2 > (i-3)c_1^2 \\ &\iff c_2 - c_1 > \sqrt{\frac{i-3}{3(i-1)}}c_1 \qquad (\text{since } c_2 \geq c_1) \\ &\iff c_2 > \left(1 + \sqrt{\frac{i-3}{3(i-1)}}\right)c_1.\end{aligned}$$

Since the above must be true for all $i = 3, \ldots, n$, we must have

$$c_2 > \max_{3 \leq i \leq n}\left(1 + \sqrt{\frac{i-3}{3(i-1)}}\right)c_1 = \left(1 + \sqrt{\frac{n-3}{3(n-1)}}\right)c_n. \tag{A.1}$$

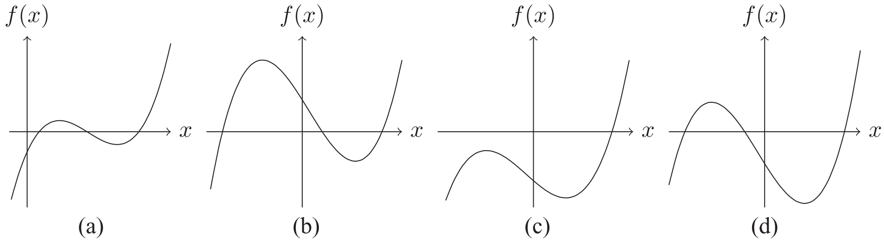

Figure A1 Different cases of the cubic function of $f(x)$.

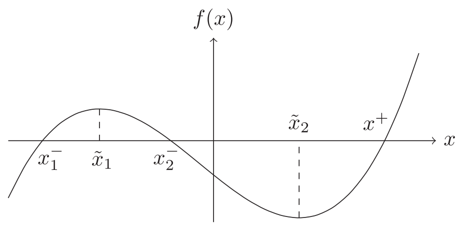

Figure A2 Cubic curves $f(x)$ with one positive root and $f(0) \leq 0$.

But

$$c_2 \geq 2c_1 > \left(1 + \sqrt{\frac{n-3}{3(n-1)}}\right) c_1,$$

thus satisfying the inequality (A.1). This completes the proof of Result 1. □

Result 2. The cubic $f(x)$ has one positive and one nonpositive critical points if $c_2 \geq 2c_1$ corresponding to the last three cubic curves in Fig. A1.

Proof of Result 2. Let x_1 and x_2 denote the two roots of the quadratic $f'(x) = 0$. Then we have

$$x_1 + x_2 = \frac{2}{3} i c_1 > 0 \text{ and } x_1 x_2 = \frac{1}{6} i(i-1) c_2 (2c_1 - c_2).$$

Since $x_1 + x_2 > 0$, at least one of the roots must be positive. If $c_2 \geq 2c_1$ then $x_1 x_2 \leq 0$ and so one root must be positive and the other must be nonpositive. Note that if $c_2 < 2c_1$ then $x_1 x_2 > 0$ and so both roots must be positive, a case that we have excluded. □

Result 3. If $c_2 \geq 2c_1$ and $s \leq 0$ then $f(x)$ has exactly one positive root corresponding to the last two cubic curves in Fig. A1.

Proof of Result 3. If $c_2 \geq 2c_1$, the cubic $f(x)$ has one nonpositive critical point $\tilde{x}_1$ and one positive critical point $\tilde{x}_2$ such that $\tilde{x}_1 \leq 0 < \tilde{x}_2$. Because the coefficient of x^3 in $f(x)$ is positive, we know that $\tilde{x}_1$ is a local maximum and $\tilde{x}_2$ is a local minimum. Because $f(0) = s \leq 0$ and the local minimum $\tilde{x}_2 > 0$, we know that $f(\tilde{x}_2) < 0$. Since $\lim_{x\to\infty} f(x) = \infty$, by using the intermediate value theorem, we conclude that a positive root $x^+ \in (\tilde{x}_2, \infty)$ exists.

If the cubic equation $f(x) = 0$ has one real root and two complex conjugate roots, it is clear that $f(x)$ has exactly one positive root. If the roots of $f(x) = 0$ are all real, the roots x_1^-, x_2^-, and x^+ satisfy that $x_1^- \leq \tilde{x}_1 \leq x_2^- < \tilde{x}_2 < x^+$ as shown in Fig. A2. Note that $\tilde{x}_2 \neq x_2^-$ and $\tilde{x}_2 \neq x^+$ because $f(\tilde{x}_2) < 0$.

Finally, we must have $x_1^- \le x_2^- \le 0$. That $x_1^- \le 0$ follows from $\tilde{x}_1 \le 0$. If $x_2^- > 0$, then for any $x \in (x_1^-, x_2^-)$, $f(x) > 0$. Since $0 \in (x_1^-, x_2^-)$ we have $f(0) > 0$, which is a contradiction. Hence we conclude that $x_2^- \le 0$. Because $x_1^- \le x_2^- \le 0$, we conclude that x^+ is the only positive root of $f(x) = 0$. □

The next result shows under what conditions is $s \le 0$.

Result 4. We have that

$$s = -\frac{i(i-1)(i-2)}{\alpha^2(n-2)}\left[\frac{1}{n(n-1)} - \frac{c_1}{n-1} - \frac{c_2^2\alpha}{2} + c_2c_1\alpha\right] \le 0$$

if and only if

$$c_1 \le c_2 \le c_1 + \sqrt{c_1^2 - \frac{2}{\alpha(n-1)}\left(c_1 - \frac{1}{n}\right)} \tag{A.2}$$

and

$$c_1 \le \frac{1}{\alpha}\sqrt{\frac{1}{n-1}}\left(\sqrt{\frac{1}{n-1}} - \sqrt{\frac{1}{n-1} - \frac{2\alpha}{n}}\right) \text{ or}$$

$$c_1 \ge \frac{1}{\alpha}\sqrt{\frac{1}{n-1}}\left(\sqrt{\frac{1}{n-1}} + \sqrt{\frac{1}{n-1} - \frac{2\alpha}{n}}\right). \tag{A.3}$$

Proof of Result 4. Note that

$$s \le 0 \iff \frac{1}{n(n-1)} - \frac{c_1}{n-1} - \frac{c_2^2\alpha}{2} + c_1c_2\alpha \ge 0 \tag{A.4}$$

$$\iff c_2^2 - 2c_1c_2 + \frac{2}{\alpha(n-1)}\left(c_1 - \frac{1}{n}\right) \le 0. \tag{A.5}$$

For this quadratic in c_2 to have real roots, its discriminant must be ≥ 0. So

$$c_1^2 - \frac{2}{\alpha(n-1)}\left(c_1 - \frac{1}{n}\right) \ge 0$$

$$\iff c_1^2 - \frac{2c_1}{\alpha(n-1)} + \frac{2}{\alpha n(n-1)} \ge 0. \tag{A.6}$$

The two roots of this quadratic inequality in c_1 are

$$\frac{1}{2}\left[\frac{2}{\alpha(n-1)} \pm \sqrt{\frac{4}{\alpha^2(n-1)^2} - \frac{8}{\alpha n(n-1)}}\right] = \frac{1}{\alpha}\sqrt{\frac{1}{n-1}}\left(\sqrt{\frac{1}{n-1}} \pm \sqrt{\frac{1}{n-1} - \frac{2\alpha}{n}}\right).$$

In the above, we have used the fact that

$$\alpha \le \frac{n}{2(n-1)} \Longrightarrow \frac{1}{n-1} - \frac{2\alpha}{n} \ge 0.$$

Furthermore, the quadratic is convex and symmetric about $c_1 = 1/\alpha(n-1) \ge n/2(n-1)$. Hence it follows that the inequality (A.6) will be satisfied if c_1 is either $\le$ the smaller root or $\ge$ the larger root.

Returning to (A.5), we see that c_2 must lie inside the interval

$$c_1 \pm \sqrt{c_1^2 - \frac{2}{\alpha(n-1)}\left(c_1 - \frac{1}{n}\right)}.$$

Since $c_2 \geq c_1$, the limits (A.2) on c_2 follow. □

Result 5. Under the three conditions stated in Theorem 1, we have $c_i \geq c_2$ for $i = 3, \ldots, n$.

Proof of Result 5. Define

$$g(x) = f(x) - s = x^3 - ic_1x^2 - i(i-1)c_2\left(\frac{c_2}{2} - c_1\right)x.$$

Then

$$\begin{aligned} g(c_2) &= c_2^3 - ic_1c_2^2 - i(i-1)\left(\frac{c_2}{2} - c_1\right)c_2^2 \\ &= c_2^2\left[i(i-2)c_1 - \left(\frac{i(i-1)}{2} - 1\right)c_2\right]. \end{aligned}$$

Note that when $i \geq 3$,

$$\frac{i(i-1)}{2} - 1 > 0.$$

Using the condition that $2c_1 \leq c_2$, we deduce

$$\begin{aligned} g(c_2) &\leq c_2^2\left[\frac{i(i-2)}{2}c_2 - \left(\frac{i(i-1)}{2} - 1\right)c_2\right] \\ &= \frac{c_2^3}{2}(2-i) \\ &< 0, \end{aligned}$$

where the last step follows from the fact that $i \geq 3$. Since $s < 0$ from Result 3, it follows that $f(c_2) = g(c_2) + s < 0$. However, $f(c_i) = 0$. Given that $f(x)$ has only one positive root, namely, $x = c_i$, it follows that $c_i > c_2$ and this is true for all $i = 3, \ldots, n$. □

We are now ready to complete the proof of Theorem 1. First we prove a lemma.

Lemma 1. *Let $h_1(x)$ and $h_2(x)$ be two continuous functions on the interval $[c, \infty)$, with continuous first derivatives. Suppose that $h_1(c) \leq h_2(c) < 0$ and $h_1'(x) \leq h_2'(x)$ for all $x \in [c, \infty)$. Further suppose that $h_1(x)$ and $h_2(x)$ have unique roots x_1^* and x_2^*, respectively. Then $x_1^* \geq x_2^*$.*

Proof of Lemma 1. First we show that $h_1(x_2^*) \leq 0$. Write

$$h_1(x_2^*) - h_1(c) = \int_c^{x_2^*} h_1'(x)dx \leq \int_c^{x_2^*} h_2'(x)dx = h_2(x_2^*) - h_2(c) = -h_2(c) \leq -h_1(c).$$

Hence $h_1(x_2^*) \leq 0$. Since there is a unique x_1^* that satisfies $h_1(x_1^*) = 0$, it follows that $x_1^* \geq x_2^*$.

Returning to the proof of the theorem, we know from Results 4 and 5 that there is a unique $c_i \in (c_2, \infty)$ such that $f_i(c_i) = 0$. Note that when $i \geq j$,

$$q_i \leq q_j \leq 0, r_i \leq r_j \leq 0 \text{ and } s_i \leq s_j \leq 0,$$

so

$$f_i(c_2) = c_2^3 + p_i c_2^2 + q_i c_2 + r_i \leq c_2^3 + p_j c_2^2 + q_j c_2 + r_j = f_j(c_2).$$

By using Result 5, $f_i(c_2) < 0$ for all $i = 3, \ldots, n$, so $f_i(c_2) \leq f_j(c_2) < 0$ if $i \geq j$.
Meanwhile, when $x \in [c_2, \infty)$,

$$f_i'(x) = 3x^2 + 2q_i x + r_i \leq 3x^2 + 2q_j x + r_j = f_j'(x).$$

By using Lemma 1, we conclude that $c_i \geq c_j$ if $i \geq j$. Therefore the monotonicity condition (3) holds.
Finally, we show that all $c_i \leq \alpha^{-2/3}$. If $c_1 = c_2 = 0$, then it is easy check that $c_n = \alpha^{-2/3}$ by substituting $i = n$ in (19). It is also easy to check that

$$f_n(\alpha^{-2/3}) = \alpha^{-2} n c_1 (1 - \alpha^{2/3}) + \alpha^{-1} n(n-1) c_2 \left(\frac{c_2}{2} - c_1\right)(1 - \alpha^{1/3}),$$

and $f_n(\alpha^{-2/3}) \geq 0$ because $c_2 \geq 2c_1$ according to Condition 3. Since $f(0) \leq 0$, we conclude that $c_n \leq \alpha^{-2/3}$ for any choice of c_1 and c_2 by the intermediate value theorem subject to the three conditions stated in Theorem 1. Because the c_is are monotone it follows that all $c_i < \alpha^{-2/3}$. This completes the proof of Theorem 1. □

Proof of Theorem 2. For third order constants, the cubic equation for c_i is given by (16) where p_i, q_i, r_i are given by (17). The second order case is a special case of the third order case when c_2 is given by (18).

First, we need to show that if $\alpha \leq n(n-1)/6(n-2)^2$ then we have $c_i \leq (i-2)/\alpha(n-2)$ for $i = 3, \ldots, n$. This is equivalent to showing that $\tilde{c}_i = c_i/(i-2) \leq 1/\alpha(n-2)$ for $i = 3, \ldots, n$. The cubic equation for $\tilde{c}_i$ is

$$\tilde{f}_i(x) = x^3 + \tilde{q}_i x^2 + \tilde{r}_i x + \tilde{s}_i = 0, \tag{A.7}$$

where $\tilde{q}_i = q_i/(i-2)$, $\tilde{r}_i = r_i/(i-2)^2$, and $\tilde{s}_i = s_i/(i-2)^3$. Using the formulae (17), it is easy to check that $\tilde{q}_i, \tilde{r}_i$, and $\tilde{s}_i$ are increasing functions of i for $i \geq 3$. Under the conditions in Theorem 1, by following a similar argument, we conclude that $\tilde{f}_i(x) = 0$ has a unique positive root $\tilde{c}_i$.

If $i \geq j$, then $\tilde{q}_j \leq \tilde{q}_i \leq 0$, $\tilde{r}_j \leq \tilde{r}_i \leq 0$, and $\tilde{s}_j \leq \tilde{s}_i \leq 0$. We have $\tilde{f}_j(0) = \tilde{s}_j \leq \tilde{s}_i = \tilde{f}_i(0) \leq 0$, and $\tilde{f}_j'(x) = 3x^2 + 2\tilde{q}_j x + \tilde{r}_j \leq 3x^2 + 2\tilde{q}_i x + \tilde{r}_i = \tilde{f}_i'(x)$ for any $x \geq 0$. By using the Lemma 1, we conclude that the positive root $\tilde{c}_i \leq \tilde{c}_j$.

Because $\tilde{c}_i \leq \tilde{c}_j$, in order to show that $\tilde{c}_i \leq 1/\alpha(n-2)$ for $i \geq 3$, we only need to show this result for $i = 3$, that is, that $\tilde{c}_3 = c_3 \leq 1/\alpha(n-2)$.

Note that

$$c_3 \leq \frac{1}{\alpha(n-2)} \Longleftrightarrow f_3\left(\frac{1}{\alpha(n-2)}\right) \geq 0.$$

Now,

$$f_3\left(\frac{1}{\alpha(n-2)}\right) = \frac{1}{\alpha^3(n-2)^3} - \frac{3c_1}{\alpha^2(n-2)^2} - \frac{6c_2}{\alpha(n-2)}\left(\frac{c_2}{2} - c_1\right)$$

$$
\begin{aligned}
& - \frac{6}{\alpha^2(n-2)}\left[\frac{1}{n(n-1)} - \frac{c_1}{n-1} - \frac{c_2^2\alpha}{2} + c_1c_2\alpha\right] \\
&= \frac{1}{\alpha^3(n-2)^3} - \frac{3c_1}{\alpha^2(n-2)^2} - \frac{6}{\alpha^2(n-2)(n-1)}\left(\frac{1}{n} - c_1\right) \\
&= \frac{1}{\alpha^2(n-2)}\left[\frac{1}{\alpha(n-2)^2} - \frac{6}{(n-1)n} + 3c_1\left(\frac{2}{n-1} - \frac{1}{n-2}\right)\right] \\
&\geq \frac{1}{\alpha^3(n-2)}\left[\frac{1}{(n-2)^2} - \frac{6\alpha}{(n-1)n}\right] \\
&\geq 0,
\end{aligned}
$$

because $2/(n-1) \geq 1/(n-2)$ when $n \geq 3$ and $\alpha \leq n(n-1)/6(n-2)^2$. We conclude that $c_i \leq (i-2)/\alpha(n-2)$ for any $i = 3, \ldots, n$.

Next, let $c_2^* \leq c_2^{**}$ denote two values of c_2 satisfying (18). Further denote the cubic function based on (c_1, c_2^*) as $f_i^*(x)$ and the cubic function based on (c_1, c_2^{**}) as $f_i^{**}(x)$. Then for any $0 \leq x \leq (i-2)/\alpha(n-2)$, we have

$$
\begin{aligned}
f_i^{**}(x) - f_i^*(x) &= x^3 - ic_1x^2 - i(i-1)c_2^{**}\left(\frac{c_2^{**}}{2} - c_1\right)x \\
&\quad - \frac{i(i-1)(i-2)}{\alpha^2(n-2)}\left[\frac{1}{n(n-1)} - \frac{c_1}{n-1} - \frac{(c_2^{**})^2\alpha}{2} + c_1c_2^{**}\alpha\right] \\
&\quad - x^3 + ic_1x^2 + i(i-1)c_2^*\left(\frac{c_2^*}{2} - c_1\right)x \\
&\quad + \frac{i(i-1)(i-2)}{\alpha^2(n-2)}\left[\frac{1}{n(n-1)} - \frac{c_1}{n-1} - \frac{(c_2^*)^2\alpha}{2} + c_1c_2^*\alpha\right] \\
&= c_2^{**}\left(\frac{c_2^{**}}{2} - c_1\right)i(i-1)\left(\frac{i-2}{\alpha(n-2)} - x\right) \\
&\quad - c_2^*\left(\frac{c_2^*}{2} - c_1\right)i(i-1)\left(\frac{i-2}{\alpha(n-2)} - x\right) \\
&= \left[c_2^{**}\left(\frac{c_2^{**}}{2} - c_1\right) - c_2^*\left(\frac{c_2^*}{2} - c_1\right)\right]i(i-1)\left(\frac{i-2}{\alpha(n-2)} - x\right) \\
&\geq 0.
\end{aligned}
$$

Note that the unique positive root c_i of the cubic equation $f_i(x) = 0$ is in $[0, (i-2)/\alpha(n-2)]$, and $f_i^{**}(x) \geq f_i^*(x)$ on the range $[0, (i-2)/\{\alpha(n-2)\}]$, so we conclude that the root c_i^{**} is less than the root c_i^*.

Next, let $c_1, c_2, \ldots, c_n$ denote the third order constants and $c_1', c_2', \ldots, c_n'$ denote the second order constants such that $c_1 = c_1'$. Note that c_2' given by (18) is the upper limit of the interval (20), so $c_2 < c_2'$. As noted just before Theorem 1, if we choose $c_2 = c_2'$ then the third order constants are the same as the second order constants, that is, $c_i = c_i'$ for $i = 1, \ldots, n$. But, since $c_2 < c_2'$, and since the c_i' are decreasing functions of c_2, it follows that $c_i > c_i'$ for $i = 3, \ldots, n$. □

Proof of Theorem 3. First note that $A_n(i) = \alpha\delta(i, n)A_{n-1}(i-1)$, we have

$$\alpha = 1 - A_n(0) = \sum_{i=1}^{n} A_n(i) = \alpha \sum_{i=1}^{n} \delta(i, n)A_{n-1}(i-1)$$

$$= \gamma_1 + \alpha \sum_{i=1}^{n} \beta_1(i-1)A_{n-1}(i-1) = \gamma_1 + \alpha \sum_{i=0}^{n-1} \beta_1(i)A_{n-1}(i).$$

Now assume the induction hypothesis that $\alpha = \gamma_k + \alpha^k \sum_{i=0}^{n-k} \beta_k(i)A_{n-k}(i)$. Then we have

$$\alpha = \gamma_k + \alpha^k \sum_{i=0}^{n-k} \beta_k(i)A_{n-k}(i)$$

$$= \gamma_k + \alpha^k \beta_k(0)A_{n-k}(0) + \alpha^k \sum_{i=1}^{n-k} \beta_k(i)A_{n-k}(i)$$

$$= \gamma_k + \alpha^k \beta_k(0) \left(1 - \sum_{i=1}^{n-k} A_{n-k}(i)\right) + \alpha^k \sum_{i=1}^{n-k} \beta_k(i)A_{n-k}(i)$$

$$= \gamma_k + \alpha^k \beta_k(0) + \alpha^k \sum_{i=1}^{n-k} (\beta_k(i) - \beta_k(0))A_{n-k}(i)$$

$$= \gamma_{k+1} + \alpha^{k+1} \sum_{i=1}^{n-k} (\beta_k(i) - \beta_k(0))\delta(i, n-k)A_{n-k-1}(i-1)$$

$$= \gamma_{k+1} + \alpha^{k+1} \sum_{i=0}^{n-k-1} (\beta_k(i+1) - \beta_k(0))\delta(i+1, n-k)A_{n-k-1}(i)$$

$$= \gamma_{k+1} + \alpha^{k+1} \sum_{i=0}^{n-k-1} \beta_{k+1}(i)A_{n-k-1}(i).$$

Thus assuming the induction hypothesis for k, we have shown it to be true for $k + 1$.

Second, by setting $\beta_k(i) = \beta_k$ as a constant and noting that $\sum_{i=0}^{n-k} A_{n-k}(i) = 1$ yields

$$\beta_k = \frac{\alpha - \gamma_k}{\alpha^k}.$$

Substituting for β_k we obtain the polynomial equation for c_{k+i}

$$\beta_k(i) + \frac{\alpha - \gamma_k}{\alpha^k} = 0.$$

It is clear that this polynomial equation has degree k by using induction on k to check the degree of $\beta_k(i)$. □

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